Lesson 32 - Lagrange Multipliers II
Applications

Last class, we learned how to use Lagrange Multipliers to find extrema (maxima and minima) of a function of two variables on a curve.

$$\begin{array}{ll} \text{maximize (or minimize):} & z = f(x, y) \\ \text{subject to:} & g(x, y) = 0 \end{array}$$

Today we will use this test to help us solve optimization word problems.

I Strategy for Lagrange Multipliers Optimization Problems

(1) **Read** the problem. Then **read it again** for details.

(2) Write your **variables** and what they mean.

(3) Write an **objective function** and whether you are trying to **maximize** it or **minimize** it.

(4) Write the **constraint equation** specified in the problem. (This is the **subject to** part of the problem.)

(5) Use the **Lagrange Multipliers** to find places where the maximum or minimum may occur.

if you don't already have 2 f(x, y) values

(6) Test another point on the constraint equation to ensure that you have found a maximum or minimum.

(7) **Answer the question.** Read carefully to know if they want the point where the extremum occurs or if they want the extreme value (plug point into $f(x, y)$.)

II Examples

Example 1 (Problem 23 in Section 7.5 of *Applied Calculus*, 9th edition, by Hoffman and Bradley). A manufacturer has $8,000 to spend on the development and promotion of a new product. It is estimated that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, sales will be approximately

$$f(x,y) = 50x^{1/2}y^{3/2}$$

units. How much money should the manufacturer allocate to development and how much to promotion to maximize sales?

x = thousands of \$ spent on dev.

y = " " " " on prom

max $f(x,y) = 50x^{1/2}y^{3/2}$

s.t. $x + y = 8$

$g(x,y) = x + y - 8 = 0$

$$\begin{cases} \dfrac{25y^{3/2}}{x^{1/2}} = \lambda \cdot 1 \\ 75x^{1/2}y^{1/2} = \lambda \cdot 1 \\ x + y = 8 \end{cases} \rightarrow \begin{cases} \dfrac{25^2y^3}{x} = \lambda^2 & (1) \\ 75^2xy = \lambda^2 & (2) \\ x + y = 8 & (3) \end{cases}$$

Equate λ^2

$$\frac{25^2y^3}{x} = 75^2xy$$

Case 1: What if $x = 0$?

$0 + y = 8$

$y = 8$

Pt: $(0,8)$ $f(0,8) = 0$

Case 2: What if $x \neq 0$?

$$\frac{25^2y^3}{x} = 75^2xy$$

$$25^2y^3 = 75^2x^2y$$

$$25^2y^3 - 75^2x^2y = 0$$

$$25^2y(y^2 - 3^2x^2) = 0$$

$$25^2y(y+3x)(y-3x) = 0$$

3 possibilities

$y = 0$ $\quad y = -3x$ $\quad y = 3x$

$y = 0$

$x + y = 8$

$x + 0 = 8$

$x = 8$

$f(8,0) = 0$

$y = \cancel{-3x}$

This would mean one of x or y is negative. Does not make sense

$y = 3x$

$x + y = 8$

$x + 3x = 8$

$4x = 8$

$x = 2$

$x + y = 8$

$2 + y = 8$

$y = 6$

$(2,6)$

$f(2,6) = 50\sqrt{2}\sqrt{6^3} > 0$

max @ $(2,6)$

<u>Ans:</u> Spend \$2000 on development
\$6000 on promotion
to maximize sales

Example 2 (Based on a problem on a LON-CAPA problem). The temperature at a point (x, y) on a plate is given by

$$T(x, y) = 3x^2 + 2y^2 - 18x + 16y$$

degrees Celsius. An ant travels in a circle on the plate that has center at $(3, -4)$ and radius 5. What is the hottest temperature encountered by the ant? The coldest temperature?

max/min $T(x,y) = 3x^2 + 2y^2 - 18x + 16y$

s.t. $(x-3)^2 + (y+4)^2 = 25$

$g(x,y) = (x-3)^2 + (y+4)^2 - 25$

$$\begin{cases} 6x - 18 = \lambda 2(x-3) & (1) \\ 4y + 16 = \lambda 2(y+4) & (2) \\ (x-3)^2 + (y+4)^2 = 25 & (3) \end{cases}$$

(1) $6x - 18 = \lambda \cdot 2(x-3)$

$3x - 9 = \lambda(x-3)$

$3x - 9 - \lambda(x-3) = 0$

$3(x-3) - \lambda(x-3) = 0$

$(x-3)(3-\lambda) = 0$

$x = 3$ OR $\lambda = 3$

$x = 3$

$\hookrightarrow$ (3)

$0^2 + (y+4)^2 = 25$

$y + 4 = \pm 5$

$y = -4 \pm 5$

$y = 1$ OR -9

Pts: $(3, 1)$

$(3, -9)$

$\lambda = 3$

$\hookrightarrow$ (2)

$4y + 16 = 2(y+4) \cdot 3$

$4y + 16 = 6y + 24$

$-8 = 2y$

$y = -4$

$\hookrightarrow$ (3)

$(x-3)^2 + 0^2 = 25$

$(x-3)^2 = 25$

$x - 3 = \pm 5$

$x = 3 \pm 5$

$x = -2, 8$

$(-2, -4)$

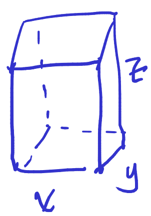
$(8, -4)$

Pts	$T(x,y) = 3x^2 + 2y^2 - 18x + 16y$
(3, 1)	-9
(3, -9)	-9
(-2, -4)	16
(8, -4)	16

highest temp: 16°C

lowest temp: -9°C

Example 3 (Problem 35 in Section 7.5 of *Applied Calculus*, 9th edition, by Hoffman and Bradley)**.** A rectangular storage shed with a flat roof is to be constructed of material that costs \$15 per square foot for the roof, \$12 per square foot for the two sides and back, and \$20 per square foot for the front. Suppose that you wanted to find the dimensions of the shed of largest volume that could be constructed for \$8000. Write down the system of equations that you would need to solve if you were using Lagrange multipliers for this problem. **DO NOT SOLVE the system.**



$$\max \; V(x,y,z) = xyz$$

$$\text{s.t.} \quad 8000 = 15xy + 12(2yz + xz) + 20xz$$

$$8000 = 15xy + 24yz + 32xz$$

$$\begin{cases} yz = (15y + 32z)\lambda \\ xz = (15x + 24z)\lambda \\ xy = (24y + 32x)\lambda \\ 8000 = 15xy + 24yz + 32xz \end{cases}$$